MATH 141 Spring 2016 Final exam practice problems – solutions

1. (a) When we plug in $x = 0$, we get $0/0$, so we use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(1-\cos x)} = \lim_{x\to 0} \frac{2x}{\sin x}.$$

Plugging in $x = 0$ gives us $0/0$ again, so we apply L'Hospital's Rule a second time to get

$$\lim_{x\to 0} \frac{2}{\cos x}.$$

Now plugging in $x = 0$ gives $2/1 = 2$.

(b) Plugging in $x = \infty$ gives $\infty \cdot \tan(3/\infty) = \infty \cdot \tan(0) = \infty \cdot 0$. This is an indeterminate form, so we cannot write the answer directly. We also cannot use L'Hospital's Rule yet – we need $0/0$ or ∞/∞ for that. So we start by rewriting the function as a quotient, using the fact that $x = \frac{1}{1/x}$:

$$x\tan(3/x) = \frac{\tan(3/x)}{1/x}.$$

So what we have is

$$\lim_{x\to\infty} \frac{\tan(3/x)}{1/x},$$

and when we plug in $x = \infty$, we get $0/0$. Now we can apply L'Hospital's Rule and take the derivative of the top and bottom to get

$$\lim_{x\to\infty} \frac{\sec^2(3/x)\cdot(-3/x^2)}{-1/x^2}.$$

We could plug in $x = \infty$ at this stage, but our limit is kind of messy. Let's clean it up by multiplying top and bottom by x^2:

$$\lim_{x\to\infty} \frac{\sec^2(3/x)\cdot(-3)}{-1} = \lim_{x\to\infty} 3\sec^2(3/x) = \lim_{x\to\infty} \frac{3}{\cos^2(3/x)}.$$

Now if we plug in $x = \infty$, we get $3/\cos^2(0) = 3/(1^2) = 3$.

(c) Plugging in $x = \infty$ gives 1^∞, which is an indeterminate form. As before, our goal is to somehow write the limit as a quotient, so that we can use L'Hospital's Rule. Let's set $y = \left(1+\frac{5}{x}\right)^x$, so that the problem is asking us to find $\lim_{x\to\infty} y$. What we do is to work with $\ln y$ for a bit:

$$\ln y = \ln\left(\left(1+\frac{5}{x}\right)^x\right) = x\ln\left(1+\frac{5}{x}\right),$$

where the last equality follows from the property of logarithms that $\ln(a^b) = b\ln a$. Now,

$$\lim_{x\to\infty} x\ln\left(1+\frac{5}{x}\right) = \infty\cdot\ln 1 = \infty\cdot 0,$$

and now we have a problem similar to the last one. We bring the x to the bottom as $1/x$ and get

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{5}{x}\right)}{1/x},$$

which gives us 0/0. Taking the derivative of top and bottom yields

$$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{5}{x}} \cdot (-5/x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{5}{1+\frac{5}{x}} = 5.$$

But be careful! 5 is not our final answer; what we found was that

$$\lim_{x\to\infty} \ln y = 5.$$

We are trying to find

$$\lim_{x\to\infty} y.$$

But now that is simple:

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^5,$$

since $\ln y$ approaches 5.

(d) As $x \to \infty$, we get ∞/∞, and so we apply L'Hospital's Rule. That gets us

$$\lim_{x\to\infty} \frac{3e^{3x} - 1}{2e^{2x} - 1}.$$

This again gets us ∞/∞, so we apply L'Hospital's Rule again:

$$\lim_{x\to\infty} \frac{9e^{3x}}{4e^{2x}}.$$

We still get ∞/∞. However, if we keep applying L'Hospital's Rule, that is going to keep happening since neither of the exponentials will go away. Instead, we simplify algebraically:

$$\lim_{x\to\infty} \frac{9e^{3x}}{4e^{2x}} = \lim_{x\to\infty} \frac{9e^{2x}e^x}{4e^{2x}} = \lim_{x\to\infty} \frac{9e^x}{4}.$$

So now as $x \to \infty$, we get an answer of ∞.

(e) As $x \to \infty$, we get $\infty - \infty$, which is indeterminate. Our goal is to change this into a product or quotient. Using the property of logarithms that $\ln f - \ln g = \ln(f/g)$, we can rewrite our limit as

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = \lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right).$$

As $x \to \infty$, the inside approaches 1, and so the answer is $\ln 1 = 0$.

(f) As $x \to 0^+$, $\ln x \to -\infty$, and so we get the indeterminate form $0 \cdot (-\infty)$. In order to use L'Hospital's Rule, we must first convert the function to an indeterminate quotient. We may rewrite $x^2 \ln x$ as $(\ln x)/x^{-2}$, which now goes to $-\infty/\infty$ as $x \to 0^+$. Applying L'Hospital's Rule now gives

$$\lim_{x\to 0^+} \frac{x^{-1}}{-2x^{-3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

2. (a) We start by using the trig identity $\sin^2 x + \cos^2 x = 1$:

$$\int \sin^5 x\, dx = \int \sin x(\sin^2 x)(\sin^2 x)\, dx = \int \sin x(1 - \cos^2 x)^2\, dx.$$

Now, we use the substitution $u = \cos x$, so that $du = -\sin x dx$. Then our integral becomes

$$-\int (1-u^2)^2 \, du.$$

Finally, we multiply out the inside, integrate, and then put back in terms of x:

$$-\int (1-u^2)^2 \, du = -\int (1-2u^2+u^4) \, du = -u + 2u^3/3 - u^5/5 = -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x$$

(b) The fact that we have a quadratic under a square root makes a trig substitution appealing. But first we need to get rid of that middle term by completing the square. Let's negate the inside of the square root to make it a bit easier to work with: $-(x^2+6x-7)$. Completing the square means taking a quadratic of the form

$$x^2 + bx + c$$

and writing it as

$$\left(x + \frac{b}{2}\right)^2 + \left(c - \left(\frac{b}{2}\right)^2\right).$$

In our case, $b = 6$ and $c = -7$, and we get

$$x^2 + 6x - 7 = (x+3)^2 - 16.$$

Therefore,

$$-(x^2 + 6x - 7) = 16 - (x+3)^2,$$

so we can rewrite our integral as

$$\int \frac{1}{\sqrt{16-(x+3)^2}} \, dx.$$

Next we make the substitution $u = x + 3$ (so $du = dx$), and we get

$$\int \frac{1}{\sqrt{16-u^2}} \, du.$$

Now we are ready to do the trig substitution. When you have $\sqrt{a^2-u^2}$, the substitution to make is $u = a\sin\theta$, which gives $du = a\cos\theta d\theta$, and we get

$$\begin{aligned}\int \frac{4\cos\theta}{\sqrt{16-16\sin^2\theta}} \, d\theta &= \int \frac{4\cos\theta}{\sqrt{16\cos^2\theta}} \, d\theta \\ &= \int \frac{4\cos\theta}{4\cos\theta} \, d\theta \\ &= \int d\theta \\ &= \theta.\end{aligned}$$

Now we undo the substitutions. Since $u = 4\sin\theta$, it follows that $\theta = \arcsin(u/4)$. Then, since $u = x + 3$, we get a final answer of $\arcsin(\frac{x+3}{4}) + C$.

(c) We use partial fractions. We want to solve the equation

$$\frac{5x^2+3x+7}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}.$$

Multiplying through both sides by the denominator on the left, we get

$$5x^2+3x+7 = A(x^2+2) + (Bx+C)(x+1).$$

Plugging in $x=-1$ yields

$$9 = 3A + 0,$$

and so $A = 3$. Plugging in $x = 0$ yields

$$7 = 2A + C,$$

and since $A = 3$, we get that $C = 1$. Finally, we can plug in any other x-value we want; let's try $x = 1$:

$$15 = 3A + 2B + 2C = 3(3) + 2B + 2(1),$$

and rearranging we get that $B = 2$. So we end up with

$$\int \frac{3}{x+1} + \frac{2x+1}{x^2+2}\,dx.$$

We split up the second fraction into 2 pieces:

$$\int \frac{3}{x+1} + \frac{2x}{x^2+2} + \frac{1}{x^2+2}\,dx.$$

The first piece becomes $3\ln|x+1|$ (using the substitution $u = x+1$). The second piece becomes $\ln|x^2+2|$ (using the substitution $u = x^2+2$). The third piece becomes $(1/\sqrt{2})\arctan(x/\sqrt{2})$, using the integration formula that

$$\int \frac{1}{x^2+a^2}\,dx = (1/a)\arctan(x/a).$$

Putting it all together yields

$$3\ln|x+1| + \ln|x^2+2| + (1/\sqrt{2})\arctan(x/\sqrt{2}) + C.$$

(d) We use Integration by Parts:

$$\begin{aligned} u &= x & dv &= e^{3x}\,dx \\ du &= dx & v &= (1/3)e^{3x} \end{aligned}$$

(To go from dv to v, we do a substitution $y = 3x$, which gives us the extra $1/3$.) Then we get $uv - \int v\,du$, which is

$$(1/3)xe^{3x} - \int (1/3)e^{3x}\,dx.$$

Completing the last integral yields

$$\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C.$$

(e) We start with the trig substitution $t = 4\sec\theta$, which gives us $dt = 4\sec\theta\tan\theta d\theta$, and so our integral becomes

$$\int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta - 16}}\, d\theta.$$

Underneath the square root, we have $16(\sec^2\theta - 1)$, which is $16\tan^2\theta$, and we simplify $\sqrt{16\tan^2\theta}$ to $4\tan\theta$. So we now have

$$\int \frac{4\sec\theta\tan\theta}{64\sec^2\theta\tan\theta}\, d\theta = \int \frac{1}{16\sec\theta}\, d\theta = \int \frac{\cos\theta}{16}\, d\theta.$$

Integrating, we get $(\sin\theta)/16 + C$. Now, $\theta = \operatorname{arcsec}(t/4)$, and so we have $(\sin(\operatorname{arcsec}(t/4)))/16 + C$. To simplify $\sin(\operatorname{arcsec}(t/4))$, we draw a right triangle with angle θ satisfying $\sec\theta = t/4$. Since $\sec = 1/\cos$, that means we want the hypotenuse to have a length of t and the adjacent leg to have a length of 4. Then the opposite leg has a length of $\sqrt{t^2 - 16}$, and so $\sin\theta = \sqrt{t^2 - 16}t$. So our final answer is

$$\frac{\sqrt{t^2 - 16}}{16t} + C$$

(f) This is a partial fractions problem; we start by trying to write

$$\frac{x^2 + 1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Multiplying through by $(x-3)(x-2)^2$ gives us

$$x^2 + 1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3).$$

Now, plugging in $x = 3$ gives us that $10 = A + 0 + 0$, so $A = 10$. Plugging in $x = 2$ gives us $5 = -C$. Now we can plug in any other x-value; let's use $x = 4$. Then we get that $17 = 4A + 2B + C$, and plugging in the values we already have for A and C gives us that $17 = 40 + 2B - 5$; therefore $B = -9$. So our integral becomes

$$\int \frac{10}{x-3} - \frac{9}{x-2} - \frac{5}{(x-2)^2}\, dx.$$

The first two pieces yield natural logs, and for the last one, we do the substitution $u = x - 2$, giving us $\int 5/u^2\, du$, which is the same as $\int 5u^{-2}\, du$, and so integrates to $-5/u$. Putting everything together, we get

$$10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C$$

3. (a) First, the integral is defined as

$$\lim_{b\to\infty}\left(\int_1^b \frac{x^2}{x^3+1}\, dx\right).$$

Since the top is the derivative of the bottom (up to a constant factor), we use the substitution $u = x^3 + 1$. That gives us $du = 3x^2 dx$, so that our integral is

$$\int \frac{1}{3u}\, du.$$

Integrating and putting back in terms of x yields $\ln(u)/3 = \ln(x^3+1)/3$. Since we wanted the definite integral from 1 to b, we substitute and subtract to get $\ln(b^3+1)/3 - \ln(1^3+1)/3$. Finally, taking the limit as $b \to \infty$ causes the inside of the natural log to go to ∞, which causes the natural log itself to go to ∞. Thus the integral diverges.

(b) We need to find

$$\lim_{b\to\infty}\left(\int_2^b \frac{1}{x^2+4}\,dx\right).$$

The antiderivative of $1/(x^2+4)$ is $\frac{1}{2}\arctan(x/2)$, and so we end up with

$$\lim_{b\to\infty}\left(\frac{1}{2}\arctan(b/2) - \frac{1}{2}\arctan(1)\right).$$

We have that $\arctan(1) = \pi/4$ (because $\tan(\pi/4) = 1$), and $\lim_{x\to\infty}\arctan x = \pi/2$ (because $\lim_{x\to\pi/2^-}\tan x = \infty$), and so we get

$$\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}.$$

(In particular, the integral converges.)

(c) Since $1/x^{1/3}$ is undefined at $x = 0$, we split the given integral into two improper integrals:

$$\int_{-1}^{8}\frac{1}{x^{1/3}}\,dx = \lim_{b\to 0^-}\left(\int_{-1}^{b}\frac{1}{x^{1/3}}\,dx\right) + \lim_{a\to 0^+}\left(\int_{a}^{8}\frac{1}{x^{1/3}}\,dx\right).$$

Since $1/x^{1/3} = x^{-1/3}$, it has antiderivative $(3/2)x^{2/3}$, and so we end up with

$$\lim_{b\to 0^-}(\frac{3}{2}b^{2/3} - \frac{3}{2}(-1)^{2/3}) + \lim_{a\to 0^+}(\frac{3}{2}8^{2/3} - \frac{3}{2}a^{2/3}).$$

Since $x^{2/3}$ is continuous, and $0^{2/3} = 0$, we get a final answer of

$$\frac{3}{2}8^{2/3} - \frac{3}{2}(-1)^{2/3} = \frac{3}{2}4 - \frac{3}{2} = \frac{9}{2}.$$

(d) The function $\tan\theta$ is undefined at $\theta = \pi/2$, and so we split the given integral into two improper integrals:

$$\int_0^{\pi}\tan\theta\,d\theta = \lim_{b\to\pi/2^-}\left(\int_0^b \tan\theta\,d\theta\right) + \lim_{a\to\pi/2^+}\left(\int_a^{\pi}\tan\theta\,d\theta\right).$$

The way to integrate $\tan\theta$ is to write it as $\sin\theta/\cos\theta$ and then make the substitution $u = \cos\theta$. This gives us $du = -\sin\theta d\theta$, and so:

$$\int\frac{\sin\theta}{\cos\theta}d\theta = \int\frac{-1}{u}du = -\ln|u| = -\ln|\cos\theta|.$$

Then the first integral becomes

$$\lim_{b\to\pi/2^-}(-\ln|\cos b| + \ln|\cos 0|).$$

As $b \to \pi/2^-$, $\cos b \to 0$, and $-\ln|\cos b| \to \infty$. So this improper integral diverges, and it follows that the entire integral diverges.

4. We need to integrate $\sqrt{(dx/dt)^2 + (dy/dt)^2}$. We have that $dx/dt = 6e^{2t}$ and $dy/dt = 6e^{3t}$. Squaring and adding gives us

$$\int_0^1 \sqrt{36e^{4t} + 36e^{6t}}\, dt.$$

Both terms under the square root are divisible by 36. In fact, they are also both divisible by e^{4t}, because $e^{6t} = e^{4t} \cdot e^{2t}$. Thus we can factor the inside of the square root to get

$$\begin{aligned}\int_0^1 \sqrt{36e^{4t} + 36e^{6t}}\, dt &= \int_0^1 \sqrt{36e^{4t}(1 + e^{2t})}\, dt \\ &= \int_0^1 \sqrt{36e^{4t}}\sqrt{1 + e^{2t}}\, dt \\ &= \int_0^1 6e^{2t}\sqrt{1 + e^{2t}}\, dt.\end{aligned}$$

Now, since the inside of the square root has its derivative sitting on the outside, we make the substitution $u = 1 + e^{2t}$, which gives us $du = 2e^{2t}dt$. This gives us

$$\int 3\sqrt{u}\, du.$$

This integrates to $3u^{3/2} * 2/3 = 2u^{3/2} = 2u\sqrt{u}$. So we get

$$2(1 + e^{2t})\sqrt{1 + e^{2t}}.$$

Substituting $t = 1$ and $t = 0$ and subtracting gives us

$$2(1 + e^2)\sqrt{1 + e^2} - 2(2)\sqrt{2}.$$

5. For these parametric equations, we have $dx/dt = \cos t - \sin t$ and $dy/dt = \cos t + \sin t$. Therefore,

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (\cos t - \sin t)^2 + (\cos t + \sin t)^2 \\ &= (\cos^2 t - 2\cos t \sin t + \sin^2 t) + (\cos^2 t + 2\cos t \sin t + \sin^2 t) \\ &= (1 - 2\cos t \sin t) + (1 + 2\cos t \sin t) \\ &= 2.\end{aligned}$$

So, to find the length, we do the integral

$$\int_0^\pi \sqrt{2}\, dt = \sqrt{2}t\ |_0^\pi = \sqrt{2}\pi.$$

6. To find the area, we need to find

$$\int_\alpha^\beta \frac{1}{2}r^2\, d\theta.$$

We start by finding the bounds. In order to get only a single petal, we want to find the θ-values that give us $r = 0$; such points will be at the origin, and if we take two such θ-values in a row, that will give us a single petal. So, we set $r = 0$ and solve for θ. If we want $3\sin(2\theta) = 0$, that means that $\sin(2\theta) = 0$. Now, $\sin x = 0$ whenever x is a multiple of π, so we want 2θ to be a multiple of π; that is, $2\theta = 0, \pi, 2\pi, \ldots$. That means that we want $\theta = 0, \pi/2, \pi, \ldots$. Any choice of two consecutive values should give us the same answer, so we will use 0 and $\pi/2$. So now we want to compute

$$\int_0^{\pi/2} \frac{1}{2}(3\sin 2\theta)^2 \, d\theta = \int_0^{\pi/2} \frac{9}{2}\sin^2(2\theta) \, d\theta.$$

To solve this, we use the half-angle formula: $\sin^2 x = (1 - \cos(2x))/2$. So we get

$$\int_0^{\pi/2} \frac{9}{4}(1 - \cos(4\theta)) \, d\theta = \frac{9}{4}(\theta - \frac{1}{4}\sin(4\theta)) \,|_0^{\pi/2} = \frac{9\pi}{8} - \frac{9}{16}(\sin(2\pi) - \sin(0)) = \frac{9\pi}{8}.$$

7. Starting at $\theta = 0$, we complete the curve when we reach $\theta = 2\pi$. So the area will be

$$\int_0^{2\pi} \frac{1}{2}(2 + \cos\theta)^2 \, d\theta.$$

Expanding out the square and using the half-angle formula, we get:

$$\begin{aligned}
\frac{1}{2}\int_0^{2\pi} (4 + 4\cos\theta + \cos^2\theta) \, d\theta &= \frac{1}{2}\int_0^{2\pi} (4 + 4\cos\theta + (\frac{1}{2} + \frac{\cos 2\theta}{2})) \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (\frac{9}{2} + 4\cos\theta + \frac{\cos 2\theta}{2}) \, d\theta \\
&= \int_0^{2\pi} (\frac{9}{4} + 2\cos\theta + \frac{\cos 2\theta}{4}) \, d\theta \\
&= \frac{9\theta}{4} + 2\sin\theta + \frac{\sin 2\theta}{8} \,|_0^{2\pi} \\
&= \frac{9\pi}{2}.
\end{aligned}$$

8. We can split the series up as

$$\sum_{n=1}^{\infty} \frac{2^{n+2}}{5^{2n}} + \sum_{n=1}^{\infty} \frac{3^n}{5^{2n}}.$$

Then note that $5^{2n} = (5^2)^n = 25^n$, and so we have

$$\sum_{n=1}^{\infty} \frac{2^{n+2}}{25^n} + \sum_{n=1}^{\infty} \frac{3^n}{25^n}.$$

Since $2^{n+2} = 2^n \cdot 2^2$, we can simplify the first one further to get

$$\sum_{n=1}^{\infty} 4 \cdot \frac{2^n}{25^n} + \sum_{n=1}^{\infty} \frac{3^n}{25^n}.$$

Both of those pieces are geometric series, and if you have a geometric series with first term a and common ratio r, then as long as $|r| < 1$, the series adds up to $\frac{a}{1-r}$. To find the first term of each series, you just plug in $n = 1$. The first series has first term $8/25$ and common ratio $2/25$, so it adds up to

$$\frac{8/25}{1 - (2/25)} = \frac{8/25}{23/5} = 8/23.$$

The second series has first term 3/25 and common ratio 3/25, so it adds up to

$$\frac{3/25}{1-(3/25)} = \frac{3/25}{22/25} = 3/22.$$

So the overall sum is $(8/23) + (3/22)$.

9. We can rewrite the sum as $\sum_{n=0}^{\infty} \left(\frac{-\pi}{4}\right)^n$. Thus, this is a geometric series with $r = -\pi/4$ and first term 1. Since $|r| < 1$, the sum of this series is

$$\frac{1}{1-(-\pi/4)} = \frac{1}{1+\pi/4} = \frac{1}{(4+\pi)/4} = \frac{4}{4+\pi}.$$

10. Listing out the first few terms, we get

$$(1 - 1/27) + (1/8 - 1/64) + (1/27 - 1/125) + (1/64 - 1/216) + \cdots .$$

If we stop there and calculate s_4, we get

$$s_4 = 1 + 1/8 - 1/125 - 1/216.$$

Adding the next term of $(1/125 - 1/343)$ gets us

$$s_5 = 1 + 1/8 - 1/216 - 1/343.$$

In general, it looks like everything cancels except for two terms at the front and two in the back. We get

$$s_n = 1 + 1/8 - 1/(n + 1)^3 - 1/(n + 2)^3.$$

Thus, the value of the series is

$$\lim_{n\to\infty}(1 + 1/8 - 1/(n + 1)^3 - 1/(n + 2)^3) = 1 + 1/8 = 9/8.$$

11. Let us write out several terms:

$$(1 - \frac{1}{\sqrt{3}}) + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}}) + (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}) + (\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}) + \cdots$$

We notice that many of the terms cancel. In general, if we write out the first n terms, we get:

$$s_n = (1-\frac{1}{\sqrt{3}})+(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{4}})+\cdots+(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n+1}})+(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+2}}) = 1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n+2}}.$$

So this is a rare case when we have an expression for s_n that doesnt include any $\cdots$ in the middle. Whenever that happens, we can just consider $\lim_{n\to\infty} s_n$, and we get that

$$\lim_{n\to\infty}(1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}) = 1 + \frac{1}{\sqrt{2}}.$$

So the series converges to $1 + (1/\sqrt{2})$.

12. (a) We use the Root Test. We have that

$$\sqrt[n]{\frac{(2n+1)^n}{n^{2n}}} = \frac{2n+1}{n^2},$$

and now we take the limit as $n \to \infty$ to get 0 (after applying L'Hospital's Rule). Since we get a limit that is less than 1, the series converges by the Root Test.

(b) If we dropped the 1 on bottom, we would have $1/n\sqrt{n^2} = 1/n^2$, and so our series looks a lot like a convergent p-series. Since

$$\frac{1}{n\sqrt{n^2+1}} < \frac{1}{n\sqrt{n^2}} = \frac{1}{n^2},$$

we see that our series is smaller than a convergent series, and so it converges by the Direct Comparison Test. (It is possible to do this with the Limit Comparison Test as well, but the argument is more involved.)

(c) This is an alternating series, so we can apply the Alternating Series Test. We set $b_n = 1/\sqrt{n+5}$ and then we need to check two things. First of all, what is $\lim_{n\to\infty} b_n$? Since the top stays the same while the bottom grows larger and larger, the limit is 0. Second of all, is it true that $b_{n+1} \leq b_n$ for all n? In other words, is $1/\sqrt{n+6} \leq 1/\sqrt{n+5}$? The answer is clearly yes, since $\sqrt{n+6} > \sqrt{n+5}$. So our series passes both parts of the Alternating Series Test, and it converges.

(d) Whenever you have one polynomial divided by another, you can get a pretty good idea of what the series does by keeping only the highest order part on top and bottom. That would give us $\sum 3n^2/10n^2$, which is the same as $\sum 3/10$. That suggests that our terms are approaching $3/10$, and as a way of checking, we use the Test for Divergence:

$$\lim_{n\to\infty} \frac{3n^2+4}{10n^2+1} = \lim_{n\to\infty} \frac{6n}{20n} = \lim_{n\to\infty} \frac{6}{20} = \frac{6}{20} \neq 0.$$

Thus, the series diverges by the Test for Divergence.

(e) The presence of factorials alerts us that the Ratio Test should work well. We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n+1)!}{(n+2)!} \cdot \frac{(n+1)!}{n!n!},$$

and using the fact that $(n+1)! = n! \cdot (n+1)$, we get that this equals

$$\frac{n!(n+1)n!(n+1)(n+1)!}{(n+2)!n!n!} = \frac{(n+1)^2(n+1)!}{(n+2)!} = \frac{(n+1)^2(n+1)!}{(n+1)!(n+2)} = \frac{(n+1)^2}{n+2}.$$

The limit of this as $n \to \infty$ is ∞ (by L'Hospital's Rule), and since we get a limit greater than 1, the Ratio Test tells us that the series diverges.

(f) The terms are all positive and top and bottom are polynomials. We try keeping highest order on top and bottom. So $a_n = \frac{n}{n^3-2}$ and $b_n = \frac{n}{n^3} = \frac{1}{n^2}$. We note that $\sum b_n$ converges (p-series, $p > 1$). We have that $a_n > b_n$, so we are "bigger than small", which tells us nothing. So we try the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3-2} = 1.$$

(To find that limit at the end, you can apply LHospitals rule or simply keep the coefficients of the n^3 on top and bottom.) Since we got a limit that was finite and nonzero, that means that $\sum a_n$ and $\sum b_n$ both have to do the same thing. Since we already know that $\sum b_n$ converges, it follows that $\sum a_n$ converges as well.

13. (a) To find the interval of convergence of a power series, you always start by applying the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{10^{n+1}x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{10^n\cdot 10\cdot x^n\cdot x}{(n+1)^3}\cdot\frac{n^3}{10^n x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{10xn^3}{(n+1)^3}\right| \\
&= |10x|\lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right| \\
&= |10x|,
\end{aligned}$$

where the last step follows by applying L'Hospital's Rule to the limit. We found that our limit, L, is $|10x|$. To find the interval, we start by setting $L = 1$ and solving. This gives us that $|10x| = 1$, which means either $10x = 1$ or $10x = -1$. So if $x = 1/10$ or $x = -1/10$, we have a limit of 1. Furthermore, if x is strictly in between $-1/10$ and $1/10$, then $L < 1$ and the series converges by the Ratio Test. To determine if the series converges at $-1/10$ or $1/10$, we plug each one in and then test the series. When $x = 1/10$, then we get

$$\sum_{n=1}^{\infty}\frac{10^n(1/10)^n}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^3},$$

which converges since it is a p-series with $p > 1$. When $x = -1/10$, we get

$$\sum_{n=1}^{\infty}\frac{10^n(-1/10)^n}{n^3} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3},$$

which converges by the Alternating Series Test. So our power series converges whenever $-1/10 \le x \le 1/10$, and thus the interval of convergence is $[-1/10, 1/10]$.

(b) As in the previous problem, we start by applying the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{2(n+1)+1}\cdot\frac{2n+1}{(-1)^n(x-3)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^n\cdot(-1)\cdot(x-3)^n\cdot(x-3)}{2n+3}\cdot\frac{2n+1}{(-1)^n(x-3)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)(x-3)(2n+1)}{2n+3}\right| \\
&= |x-3|\lim_{n\to\infty}\left|\frac{-(2n+1)}{2n+3}\right| \\
&= |x-3|.
\end{aligned}$$

We get a limit of $L = |x-3|$. To find the interval of convergence, we start by setting $L = 1$ and solving. In this case, that gives us $|x-3| = 1$, which means that either $x-3 = 1$ or $x-3 = -1$. The first yields $x = 4$ and the second yields $x = 2$. If $2 < x < 4$, then $|x-3| < 1$, and the Ratio Test says that the series converges. Now we test the endpoints separately. If $x = 2$, then our series becomes

$$\sum_{n=0}^{\infty}(-1)^n \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1},$$

which diverges (use a Limit Comparison Test on $1/n$). If $x = 4$, then our series becomes

$$\sum_{n=0}^{\infty}(-1)^n \frac{(1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

which converges by the Alternating Series Test. So our interval of convergence is $(2, 4]$.

14. The Maclaurin series for $\cos(x)$ is

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots.$$

To get the Maclaurin series for $\cos(x^2)$, we simply change every x to x^2 in the above, which changes x^{2n} to $(x^2)^{2n} = x^{4n}$. We get:

$$\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!} = 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \cdots.$$

Then, multiplying through by x^3 yields:

$$x^3\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n)!} = x^3 - \frac{x^7}{2} + \frac{x^{11}}{24} - \frac{x^{15}}{720} + \cdots.$$

To integrate that, we just integrate each term separately. We get the sigma notation for the new series by just integrating the inside of the existing sigma. So we get

$$\int x^3\cos(x^2)\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+4}}{(4n+4)(2n)!} = C + \frac{x^4}{4} - \frac{x^8}{16} + \frac{x^{12}}{12\cdot 24} - \frac{x^{16}}{16\cdot 720} + \cdots.$$

15. The Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

To get the Maclaurin series for $e^x - 1/x$, we simply subtract off the first term, then divide every term through by x:

$$\frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots.$$

To integrate that, we just integrate each term separately. We get the sigma notation for the new series by just integrating the inside of the existing sigma. So we get:

$$\int \frac{e^x - 1}{x}\, dx = C + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \cdots = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}.$$