MA480 – Introduction to Computational Algebraic Geometry

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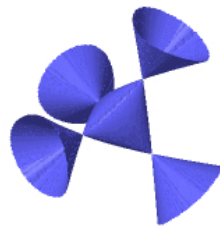**Office: Sci-3-082**
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Course Description

An *affine algebraic variety* is the solution set of a system of polynomial equations. Low-dimensional examples include ellipses, hyperbolas, and parabolas, and also the surfaces shown below:
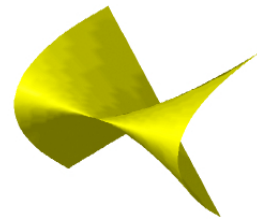

$$x^2 + y^2 - x^2 z + y^2 z + z^2 = 1$$

Cayley's surface, with four *double points* and *tetrahedral symmetry*.

$$3x^2 y^2 + 6xyz = 4x^3 z + 4y^3 + z^2$$

Tangent surface to the twisted cubic, an example of a *ruled surface* with a crease along its base curve.

$$x^2 = y^2 z$$

Whitney's umbrella, showing a *pinch point singularity*.

This course is an introduction to the geometry of affine algebraic varieties, with emphasis on two core themes:

The Algebra-Geometry Dictionary: Descartes' introduction of *coordinates* in 1637 hinted at the essential unity of algebra and geometry. Three and a half centuries of subsequent development showed the correspondence to be deeper and more complete than Descartes could have foreseen. We will describe a "dictionary" by which one can translate geometric ideas into algebraic ones, and vice-versa.

Buchberger's Algorithm: In linear algebra one has the familiar *Gauss-Jordan algorithm* for solving systems of linear equations, and by extending this algorithm in various ways one can compute (almost) everything worth knowing about the solutions of such systems. *Buchberger's algorithm*, first described in the 1960s, generalizes Gauss-Jordan elimination to systems of polynomial equations of arbitrary degree, and various extensions of

it compute almost everything worth knowing about affine algebraic varieties.[1] Because Buchberger's algorithm is computationally intensive, we will almost never carry it out by hand; instead, we shall make extensive use of **computer algebra systems** for computation and visualization.

Students who complete this course will acquire a strong foundation for the study of various applications both within and outside mathematics. In particular, our textbook contains material on **robotics**, **computer aided design**, **automatic theorem proving**, **invariant theory**, **projective geometry**, and **computer vision**. In addition, highly motivated students will be prepared to participate meaningfully in the department's **current research** in invariant theory and the geometry of nilpotent orbits.

Prerequisites

Admission to the course is contingent upon successful completion of MA260 or an equivalent linear algebra course.

Text

There is one required text for the course: *Ideals, Varieties, and Algorithms*, third edition, by David Cox, John Little, and Donal O'Shea.

Grading

Course grades are based on weekly quizzes (20%), two in-class tests (20% each), and a cumulative final exam (40%).

Reading and class preparation

There is a reading assignment associated with each class period. Although it is not generally possible to discuss every topic in class, students are responsible for the entire content of the reading assignment. *Test and exam questions may cover reading material not discussed explicitly in class.* Consequently it is very important to complete the reading assignments on time and to come to class prepared with questions.

Make-up tests

Tests may be rescheduled only in cases of serious illness, bereavement, or other circumstances of similar gravity. Whenever possible, arrangements for make-up tests must be made *in advance* of the regularly scheduled testing time.

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[1] At least in principle. Buchberger's algorithm has high computational complexity, and it is easy to make examples which overwhelm even the strongest computers.

Student conduct

Students are required to adhere to the University Policy on Academic Standards and Cheating, to the University Statement on Plagiarism and the Documentation of Written Work, and to the Code of Student Conduct as delineated in the catalog of Undergraduate Programs, pp. 44–45 and 48–52. The Code is available online at the following web site:

    http://www.umb.edu/editor_uploads/images/life_on_campus/CSC.pdf

Web page

This syllabus and other course materials are available on-line at

    http://cartan.math.umb.edu/wiki/index.php/Math_480,_Spring_2013

Course Calendar

Homework problems should be done prior to the due date but **are not to be handed in.** *One problem from each assignment will appear on the weekly quiz.*

Tuesday, January 29: Introduction.

Thursday, January 31: Polynomials and affine space.

Read before class: Section 1.1.

Tuesday, February 5: Affine varieties.

Read before class: Section 1.2.

Do before class: Assignment 1.

Thursday, February 7: Parametrizations of affine varieties.

Read before class: Section 1.3.

Tuesday, February 12: Ideals.

Read before class: Section 1.4.

Do before class: Assignment 2.

Thursday, February 14: Polynomials of one variable.

Read before class: Section 1.5.

Tuesday, February 19: Introduction to Gröbner bases.

Read before class: Section 2.1.

Do before class: Assignment 3.

Thursday, February 21: Monomial orderings.

Read before class: Section 2.2.

Tuesday, February 26: Multivariable division algorithm.

Read before class: Section 2.3.

Do before class: Assignment 4.

Thursday, February 28: Monomial ideals and Dickson's Lemma.

Read before class: Section 2.4.

Tuesday, March 5: Test 1 (sections 1.1–2.3).

Do before class: Assignment 5.

Thursday, March 7: The Hilbert Basis Theorem. Gröbner bases.

Read before class: Section 2.5.

Tuesday, March 12: Properties of Gröbner bases.

Read before class: Section 2.6.

Do before class: Assignment 6.

Thursday, March 14: Buchberger's Algorithm.

Read before class: Section 2.7.

Tuesday, March 26: First applications of Gröbner bases.

Read before class: Section 2.8.

Do before class: Assignment 7.

Thursday, March 28: The elimination and extension theorems.

Read before class: Section 3.1.

Tuesday, April 2: The geometry of elimination.

Read before class: Section 3.2.

Do before class: Assignment 8.

Thursday, April 4: Implicitization.

Read before class: Section 3.3.

Tuesday, April 9: Singular points and envelopes.

Read before class: Section 3.4.

Do before class: Assignment 9.

Thursday, April 11: Unique factorization and resultants.

Read before class: Section 3.5.

Tuesday, April 16: Test 2 (sections 2.4–3.3).

Do before class: Assignment 10.

Thursday, April 18: Resultants and the extension theorem.

Read before class: Section 3.6.

Tuesday, April 23: The Nullstellensatz.

Read before class: Section 3.7.

Do before class: Assignment 11.

Thursday, April 25: Radical ideals and the ideal-variety correspondence.

Read before class: Section 4.2.

Tuesday, April 30: Sums, products, and intersections of ideals.

Read before class: Section 4.3.

Do before class: Assignment 12.

Thursday, May 2: Zariski closure and quotients of ideals.

Read before class: Section 4.4.

Tuesday, May 7: Irreducible varieties and prime ideals.

Read before class: Section 4.5.

Do before class: Assignment 13.

Thursday, May 9: Decomposition of a variety into irreducible components.

Read before class: Section 4.6.

Tuesday, May 14: Summary. Topics for further reading.

Read before class: Section 4.8.

Do before class: Assignment 14.